

On the Theoretical Value of the Acceleration of the Moon's Mean Motion in Longitude, produced by the Change of Eccentricity of the Earth's Orbit. By Sir G. B. Airy, K.C.B. Astronomer Royal.

1. In the *Monthly Notice* of the Royal Astronomical Society for 1874, January, I explained the principles of a novel method of treating the Lunar Theory. My object was to dispense with long algebraic expansions, far as as possible. For this purpose, I proposed to use the numerical coefficients found by a preceding investigator (Delaunay), not as accurate, but as very approximate, requiring numerical corrections (treated, in the first instance, in a symbolical form) so small that multiples of those symbols (when reduced to numbers) to the first power only would suffice for the correction of the numerical value of every function depending on the coefficients. The immediate object was to find the numerical relation between the theoretical movements of the Moon and the forces (themselves depending on those movements) which ought to account for the movements; and to find whether, by varying the movements and varying in correspondence therewith the forces which depend on those movements, the relation between the movements and the forces can be made perfect.

2. It is evident that this form of investigation is not limited to the correction of assumed coefficients, but can also be applied to the examination of the effects of introducing small forces not contemplated in the original theory. These small forces may in some instances depend on geometrical considerations (as in the effect of the Earth's oblateness), in other cases they may depend on time (as in the slow change of solar elements producing certain disturbances in the motions of the Moon). In such cases, the changes from the original suppositions are in fact the introduction of new forces; and here we have to consider how, by varying the movements and varying in correspondence therewith the forces which depend on those movements (as stated in the first paragraph), and also introducing the new forces and their variations depending on the movements, the relation between the movements and the forces can be made perfect. The original fundamental supposition must, however, be maintained, that the new variations of forces and movements are to be so small that the first power of the algebraic terms expressing them will suffice.

3. The units employed in the further investigation which I am now to explain (the same which are used in my general Lunar Theory) are the following:—

The unit of longitudinal measure is the mean distance of the Moon from the Earth at the epoch (say A.D. 1900). In respect of absolute measure, that distance is called 1, but in respect of variability, it is called a . In the present investigation, we shall not need to consider the variability.

The unit of angular measure is the angle corresponding to an arc of a circle whose length is equal to radius 1. It is equal, in degrees, to $\frac{360^\circ}{2\pi}$ or, in seconds, to $\frac{1296000''}{2\pi}$.

The unit of time is the time occupied by the Moon, with her mean angular motion at the epoch, in describing the unit of angle. We shall have to compare it with the length of a Julian year: it is easily seen that, as referred to our unit, the Julian year =

$$\frac{\text{Moon's mean angular motion}}{\text{Sun's mean angular motion}} \times 2\pi = 13.368753 \times 2\pi = 83.998352.$$

The unit of the Moon's angular motion is therefore necessarily the angular velocity with which the Moon in mean angular motion describes the angle 1 in the time 1. In respect of absolute measure of angular motion, this is called 1; in respect of variability (not, however, considered here), it is called n .

4. The Moon's radius vector, the projection of radius vector on the plane of the ecliptic, her longitude, her latitude, and all combinations of these, are (in the most general theory) expressed in indefinite series of periodical terms, whose arguments consist of sums of various multiples (sometimes positive, sometimes negative) of the following quantities:—

D = excess of Moon's mean longitude above Sun's mean longitude.

f = excess of Moon's mean longitude above mean longitude of node.

l = excess of Moon's mean longitude above mean longitude of perigee.

S = mean anomaly of Sun.

These are all treated as increasing uniformly with the time.

In the present investigation, however, it is proposed to neglect the variable terms related to the excentricity of the Moon's orbit and its inclination to the ecliptic, not as non-existent, but as producing nothing of sensible amount additive to the terms employed in the present inquiry; and thus the symbols f and l will not appear. The modifications which they produce in the constant term will, in the first instance, be retained. We shall not treat of the disturbance of the Moon's latitude. As regards the excentricity of the Sun's orbit, we shall use a mean of the Sun's effect through each year; and thus S will not be required. Thus the only argument remaining is D , and the only multiple of that argument which we shall use is $2D$, of which the effects far exceed those of D or $3D$. And for $2D$ we shall write F .

5. The equations which we use in the general theory to determine the Moon's motion apply to T (a force in the plane of the Sun's orbit, normal to the projection of the Moon's radius

vector, and accelerating the Moon), P (a force in the same plane, in the projection of the radius vector, and directed from the Earth), and Z (a force normal to the plane of the Sun's orbit, in the direction in which z is considered positive). Z , however, has here no existence, as the inclination of the two orbits is neglected.

6. By investigations of which the introductory parts are printed, a "Factorial Table" has been prepared, exhibiting in a symbolical form, as factors of the lunar geometrical disturbances, the forces which must be introduced as acting on the Moon in order to explain the disturbed movements of the Moon. Any possible disturbances, subject to the limitation above mentioned in regard to inclination of orbit, may be expressed by disturbances of $\frac{a}{r}$ and v , where r is the radius vector of the Moon, and v the Moon's true longitude. For such disturbances we put the symbols $\delta \frac{a}{r}$ and δv . Then the Factorial Table, limited as is above mentioned, gives the following equations, in which the value of $\frac{r}{a}$ is $1 - .0077 \cos F$:

$$T \frac{r}{a} = \left\{ \begin{array}{ll} + \delta \left(\frac{a}{r} \right) & \times (\quad .0000 \quad - .0281 \sin F) \\ + \frac{d}{dt} \left(\delta \frac{a}{r} \right) & \times (-1.9919 \quad + .0061 \cos F) \\ + \delta v & \times (\quad .0000 \quad + .0168 \cos F) \\ + \frac{d}{dt} (\delta v) & \times (\quad .0000 \quad + .0268 \sin F) \\ + \frac{d^2}{dt^2} (\delta v) & \times (+1.0006 \quad - .0144 \cos F) \end{array} \right\}$$

$$P \frac{r}{a} = \left\{ \begin{array}{ll} + \delta \left(\frac{a}{r} \right) & \times (+2.9964 \quad - .0236 \cos F) \\ + \frac{d}{dt} \left(\delta \frac{a}{r} \right) & \times (\quad .0000 \quad - .0528 \sin F) \\ + \frac{d^2}{dt^2} \left(\delta \frac{a}{r} \right) & \times (-1.0053 \quad + .0212 \cos F) \\ + \delta v & \times (\quad .0000 \quad + .0168 \sin F) \\ + \frac{d}{dt} (\delta v) & \times (-1.9888 \quad - .0090 \cos F) \end{array} \right\}$$

These equations give, in the first instance, the values of forces which can produce an assigned geometrical disturbance; our object, however, will be, by a reversal-process, to find the geometrical disturbances which will be required to correspond to an assigned external force.

7. It is important to observe that, in forming these equations, the variations of terrestrial attraction and of solar attraction,

depending on the change of the Moon's place produced by the disturbance of which we treat, are accurately taken into account, whatever the change of place may be.

We now proceed to compute the Disturbing Forces.

8. Taking the mean parallaxes of the Sun and Moon as $8''.91$ and $3422''.3$, the following values are obtained by known developments:—

Mean tangential disturbing force at epoch $-\cdot008284 \cdot \sin F.$

Mean radial disturbing force at epoch $+\cdot002683 + \cdot008245 \cdot \cos F.$

(These numbers have been computed in a general process, embracing all the inequalities of the lunar motions.)

Now suppose that, in the progress of time, the solar energy which produces these forces varies from the force at tabular epoch in the proportion of $1 : 1 + bt$. This is the same thing as introducing new forces, whose values are equal to the values above-given, multiplied by bt . Thus we obtain—

$$T = -\cdot008284 \times bt \cdot \sin F;$$

$$P = +\cdot002683 \times bt + \cdot008245 \times bt \cdot \cos F;$$

and these quantities, multiplied by $1 - \cdot0077 \cos F$, are to be substituted on the left side of the equation in Article 6. But the multiplication of the small term by T , and by the second part of P , will produce terms depending on $2F$, which are everywhere neglected. The multiplication of the first part of P , however, introduces $-\cdot002683 \times \cdot0027 \times bt \cdot \cos F$. This is similar to the second part, and will be associated with it.

9. In future we shall adopt the following notation:—

For $\frac{dF}{dt}$ we shall write m : the numerical value of m is $+1.8504$. For

$\cdot008284$, $\cdot002683$, and $\cdot008245$ (numbers without signs),

we shall use A , B , C .

A and C are sensibly equal.

The equations of Article 6 now become—

Equation for the Moon's ecliptic longitude:—

$$-A \cdot bt \cdot \sin F = \left\{ \begin{array}{l} + \frac{\delta a}{r} \times (\cdot0000 - \cdot0281 \sin F) \\ + \frac{d}{dt} \left(\frac{\delta a}{r} \right) \times (-1.9919 + \cdot0061 \cos F) \\ + \delta v \times (\cdot0000 + \cdot0168 \cos F) \\ + \frac{d}{dt} (\delta v) \times (\cdot0000 + \cdot0268 \sin F) \\ + \frac{d^2}{dt^2} (\delta v) \times (+1.0006 - \cdot0144 \cos F) \end{array} \right\}$$

Equation for the projection of the Moon's radius vector:—

$$+ B.\delta t. \left. \begin{aligned} &+ (C - .0077.B)\delta t. \cos F \end{aligned} \right\} = \left\{ \begin{aligned} &+ \delta \frac{a}{r} \quad \times (+2.9964 - .0236 \cos F) \\ &+ \frac{d}{dt} \left(\delta \frac{a}{r} \right) \quad \times (.0000 - .0528 \sin F) \\ &+ \frac{d^2}{dt^2} \left(\delta \frac{a}{r} \right) \quad \times (-1.0053 + .0212 \cos F) \\ &+ \delta v \quad \times (.0000 + .0168 \sin F) \\ &+ \frac{d}{dt} (\delta v) \quad \times (-1.9888 - .0090 \cos F) \end{aligned} \right\}$$

In the following Articles we shall use C' for $(C - .0077.B)$, and in future calculations we shall use the integral numbers $-2, +1, +3, -1, -2$, for the broken numbers in the first column, 2nd, 5th, 6th, 8th and 10th lines.

10. It is perhaps vain to expect that equations so complex as those exhibited above can be solved directly. The only hope to solve them rapidly and successfully must be in carefully selecting a form for the solution, and in substituting its terms with indeterminate coefficients. The form adopted has been established from the following considerations:—

(1.) Whatever takes place in the Moon's mean parallax will be repeated, lunation after lunation, in the same way. Thus $\delta \frac{a}{r}$ may have a term $+ct$.

(2.) A corresponding change will take place in the mean arc of longitude described in every portion of time, so that the alteration of mean arc described in one portion, which is the sum of all alterations in the portions up to that time, will be proportional to t ; and the sum of these alterations of mean arc up to that time, which is the alteration of longitude, will be proportional to t^2 . Thus δv may have a term ht^2 .

(3.) and (4.) Since the Sun's motion is not altered, F , which = 2 Moon's longitude $-$ 2 Sun's longitude, will receive the addition $2ht^2$; and this term will geometrically (not by multiplication of any other perturbation) modify the original quadrantal terms, so that $\delta \frac{a}{r}$ will have a term such as $ft^2 \sin F$, and δv will have such a term as $kt^2 \cos F$.

(5.) and (6.) During the progress of these changes, it is not unreasonable to expect that the quadrantal modification of the orbit may undergo changes proportional to the time; and it seems likely that $\delta \frac{a}{r}$ may require such a term as $gt \cos F$ and that δv may require $lt \sin F$.

(7.) and (8.) Something will depend on the time when we suppose the intrusion of the new force and the disturbance of movements to begin. It may happen that the new force will

not begin at Lunar Conjunction or Opposition. Our assumption $T = -A b t \sin F$ implies that there will never be a disturbing force in longitude when F has one special value + integral multiples of π . It appears that this can be well met by the term $e \cdot \sin F$ in $\delta \frac{a}{r}$ and a constant term i in δv .

11. The following assumptions, therefore, have been made:—

$$\delta \frac{a}{r} = +c \cdot t + e \cdot \sin F + f t^2 \cdot \sin F + g t \cdot \cos F;$$

$$\delta v = +h t^2 + i \quad + k t^2 \cdot \cos F + l t \cdot \sin F,$$

containing eight indeterminate quantities; and these expressions and their differentials have been substituted in the equations of Article 9. It is proper to remark that in the multiplication of series forming every step of this operation, $\sin F$ and $\cos F$ are most carefully preserved; but $\sin^2 F$, $\cos^2 F$, and $\sin F \cdot \cos F$, are neglected; they all relate to the argument $2F$ or $4D$, which has been omitted from the adopted portion of the Factorial Table; and their factors are all very small.

The results of substitution admit of being arranged in eight groups, as follows:—

$$(1) \quad -2c + 2h = 0.$$

$$(2) \quad (+3c - 4h - Bb)t = 0.$$

$$(3) \quad (-2mf + \cdot 0168 \cdot h - m^2 k)t^2 \cdot \cos F = 0.$$

$$(4) \quad (+3f + m^2 f + \cdot 0168 \cdot h + 2mk)t^2 \cdot \sin F = 0.$$

$$(5) \quad (-\cdot 0281 \cdot c - 4f + 2mg + \cdot 0536 \cdot h - 4mk - m^2 l + Ab)t \cdot \sin F = 0.$$

$$(6) \quad (-\cdot 0234 \cdot c - 4mf + (m^2 + 3)g - \cdot 180 \cdot h - 4k - 2ml - C'b)t \cdot \cos F = 0.$$

$$(7) \quad (+\cdot 0061 \cdot c - 2me - 2g - \cdot 0288 \cdot h + \cdot 0168 \cdot i + 2k + 2ml) \cdot \cos F = 0.$$

$$(8) \quad (-\cdot 0528 \cdot c + (m^2 + 3)e - 2f + 2mg + \cdot 0168 \cdot i - 2l) \cdot \sin F = 0.$$

12. The solution of these equations, though troublesome, is not difficult. The following appear to be the results. In the last column, the expressions are simplified by considering $A=3B$, $C=3B$:—

$$c = \quad \quad \quad -Bb.$$

$$e = \quad \quad \quad + \cdot 6195 \cdot Bb.$$

$$f = \quad \quad \quad + \cdot 0144 \cdot Bb.$$

$$g = \quad + \cdot 4485 \cdot Ab \quad + \cdot 0238 \cdot Bb \quad + \cdot 4141 \cdot C'b. \quad = + \cdot 26116 \cdot Bb.$$

$$h = \quad \quad \quad -Bb.$$

$$i = \quad \quad \quad - \cdot 37083 \cdot Bb.$$

$$k = \quad \quad \quad - \cdot 0205 \cdot Bb.$$

$$l = \quad + \cdot 7769 \cdot Ab \quad + \cdot 0457 \cdot Bb \quad + \cdot 4476 \cdot C'b \quad = + \cdot 37192 \cdot Bb.$$

It will be remembered that the signs of c , e , f , and g , are those which apply to $\delta \left(\frac{a}{r} \right)$. The signs are to be changed for $\delta \left(\frac{r}{a} \right)$.

The term which expresses the secular acceleration in longitude is $ht^2 = -Bb.t^2$. To the numerical value of this term we will now give attention.

13. First, we have to ascertain the value, in ordinary language, of $B = .002683$. The value is here expressed in decimal parts of the radius; we require it in seconds of arc. Now

$$2\pi = 360 \times 60 \times 60 \times 1'' = 1296000 \times 1'';$$

$$\text{therefore } .002683 = \frac{1296000''}{2\pi} \times .002683.$$

14. Secondly, to find the value of b in terms of ΔE . Let σ , A , E , R , V , T , be the Sun's mass, the orbital semiaxis major, the excentricity of the solar (or terrestrial) orbit, the orbital radius vector, the Sun's true longitude, and the Sun's periodic time (one year.) The mean effect (as regards special points of the Moon's orbit) of the Sun, in a given state of his orbit and a given position in his orbit, to disturb the Moon's motion, is represented by $\frac{\sigma a}{R^3} \times \Delta t$, Δt being an element of time. Let ΔV be the corresponding element of longitude. The area described by the radius vector is $\frac{1}{2}R^2\Delta V$. The whole area of the ellipse is $\pi A^2\sqrt{(1-E^2)}$. Then this proportion holds—

$$\Delta t : T :: \frac{1}{2}R^2\Delta V : \pi A^2\sqrt{(1-E^2)};$$

$$\text{and therefore } \Delta t = \frac{TR^2 \cdot \Delta V}{2\pi A^2\sqrt{(1-E^2)}};$$

and the disturbing effect of the Sun in the element of time is

$$\frac{\sigma a}{R^3} \cdot \frac{TR^2 \cdot \Delta V}{2\pi A^2\sqrt{(1-E^2)}} = \frac{\sigma a T}{2\pi \cdot A^2\sqrt{(1-E^2)}} \cdot \frac{\Delta V}{R}.$$

Now

$$\Delta v = \Delta \text{ (true anomaly), and } R = \frac{A(1-E^2)}{1+E \cdot \cos \text{ (true anomaly) }};$$

therefore the disturbing effect

$$= \frac{\sigma a}{2\pi \cdot A^3} \times \frac{1+E \cdot \cos \text{ (true anomaly) }}{1-E^2} \times \frac{T \cdot \Delta \text{ (true anomaly) }}{\sqrt{(1-E^2)}}.$$

Integrating this through an entire orbital revolution of the Sun, the value is

$$\sigma a \times \frac{T}{A^3} \times \frac{1}{(1-E^2)^{\frac{3}{2}}} = \sigma a \cdot \frac{T}{A^3} \left\{ 1 + \frac{3}{2}E^2 + \frac{3 \cdot 5}{2 \cdot 4}E^4 + \&c. \right\}.$$

The variation of this quantity depending on the variation δE is

$$+ \sigma a \frac{T}{A^3} \times \left\{ 3E + \frac{15}{2}E^3 \right\} \delta E$$

and the proportion of the variation to the unvaried quantity is

$$+ \frac{3E + \frac{15}{2}E^3}{1 + \frac{3}{2}E^2} = +3E \times \frac{1 + \frac{5}{2}E^2}{1 + \frac{3}{2}E^2} = +3E(1 + E^2)\delta E, \text{ nearly.}$$

15. Thirdly, to find the values of E and δE , and the proportion of the variation to the unvaried quantity just mentioned:

By Le Verrier's *Annales de l'Observatoire de Paris*, tome iv., page 103, $E = .01676927 - .0000004338 \times \text{number of years}$; and the proportion of the variation above mentioned, for one year, to the unvaried quantity, is

$$-3 \times .01676927 \times \{1 + (.01677)^2\} \times .0000004338.$$

To infer from this the proportion for one orbital revolution of the Moon, we must multiply by

$$\frac{\text{time of Moon's orbital revolution}}{\text{time of Sun's orbital revolution}}, \text{ or } \frac{\text{Sun's motion in longitude for 30 days}}{\text{Moon's motion in longitude for 30 days}}$$

$$\text{or } \frac{106445.7}{1423046.6}$$

And to infer the proportion for our unit of time, we must further multiply by

$$\frac{\text{unit of time}}{\text{units of time in Moon's orbital revolution}} \text{ or } \frac{1}{2\pi}.$$

Thus we obtain for the proportion of the variation to the unvaried quantity during one unit of time, or for b ,

$$-3 \times .01676927 \times 1.0002812 \times .0000004338 \times \frac{1064457}{2\pi \times 14230466}.$$

16. And to obtain the value of δv for one year, we must multiply this by the square of $\frac{\text{time of Sun's orbital motion}}{\text{time of Moon's orbital motion}} \times$

units of time in Moon's orbital motion, or $\left(\frac{2\pi \times 14230466}{1064457}\right)^2$.

Thus we obtain finally for the value of δv or $-Bb.t^2$,

$$\text{For one year, } .000101477.$$

$$\text{For a century, } 10.1477.$$

I present this result to the Society with much confidence.

It is to be remarked that this numerical value is founded upon the numerical value attached to the solar parallax in the first step of the investigations—namely $8''.91$. And the magnitudes of the quantities A, B, C , depend upon the inverse cube of the Sun's distance, or upon the cube of parallax. If, for instance,

the parallax be diminished by $\frac{1}{60}$ part, the magnitudes of A, B, C, and of the acceleration, will be diminished by $\frac{1}{20}$ part.

17. The following deductions are unimportant, but they may be interesting. In the Moon's orbit, $1'' = 6000$ feet, very nearly. In a century, therefore, the Moon is accelerated 60000 feet. In the first year, the acceleration is 6 feet; in the second year, 18 feet additional, &c. This is additive to the computed longitude, whether before or after the epoch.

In a century, the Moon's distance is changed by $\frac{60000 \text{ feet}}{t}$.

Now t for 100 years is 8400, and the Moon's distance is changed in a century by $\frac{60000}{8400}$ feet or 7.14 feet; in one year it is changed by 0.0714 feet, or less than an inch. This change proceeds uniformly; for every year before the epoch, the distance is additionally greater than the computed distance by the multiple of 0.0714 feet, and for every year after the epoch, it is additionally less.

Wherever Bb occurs, we may use the value found thus. In a century, δv , or ht^2 , or $-Bb \cdot t^2$, or $-Bb \times (8400)^2 = 60000$ feet. Therefore Bb (without respect of sign) $= \frac{60000 \text{ feet}}{(8400)^2} = \frac{6 \text{ feet}}{(84)^2} = 0.0101 \text{ inch}$.

Royal Observatory, Greenwich,
1880, April 24.

The Nebula in the Pleiades. By A. A. Common, Esq.

An observation of this Nebula was made on February 8th, 1880, with my three-feet telescope, and the sketch made at this time, a copy of which is attached, differs so from that of Mr. Maxwell Hall in the January number of the *Monthly Notices* (which came to hand soon after) that it may be worth recording. The stars on this sketch are traced from two photographs taken that night with one and a half minute's exposure, and are given as a guide to the positions of the nebulae seen.

The oval patch of light near *Alcyone* was only seen on this night; later observations made with a view to correct the place of the nebula proper did not show it, but the nights subsequent to the first were not so fine. The smaller patch of light north of *Merope* was always seen; the edge near that star is pretty sharply defined, with dark sky between. The general shape and direction of the nebula proper was as shown. A fine night was waited for with the hope of seeing better the extent of this nebula, but without success. There were pretty certain indications of an extension beyond *Merope* in the direction of *Electra*. In apparent brightness the sharp edge of the smaller nebula was equal to the brighter part of the large one; but this may have been due